Pro Tip: Volumes and Areas are never negative.

Math 2D Morning Last Quiz - March 10th

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Show all of your work. *There is a question on the back side.

1. You will be finding the volume of the following:

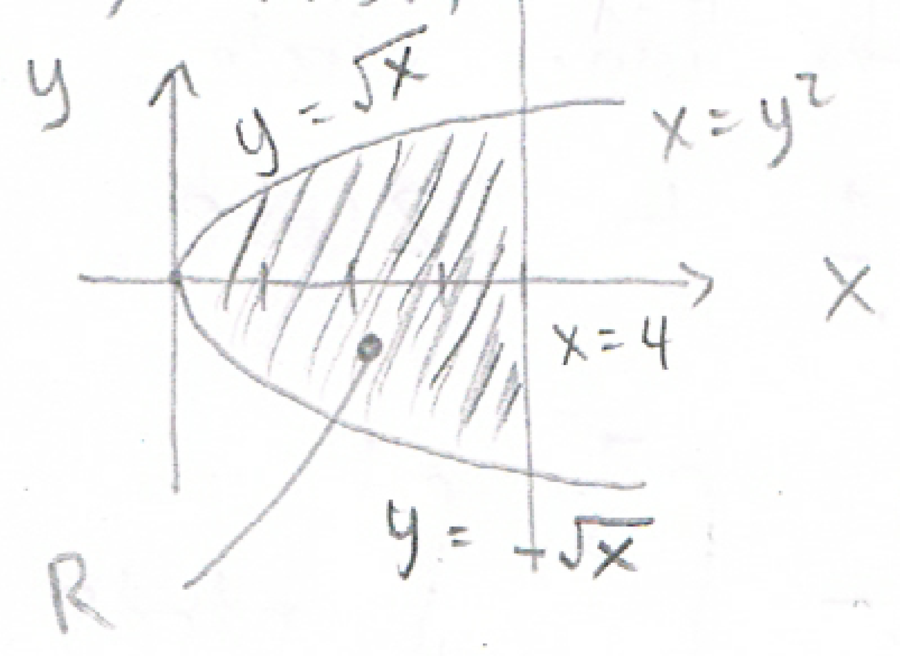
The volume between the surface $z = 1 + x^2y^2$ and the plane $z = 1$, above the region enclosed by the curves $x = y^2$ and $x = 4$.

(a) Write the integral that gives your volume in both ways, as $\iint\limits_R f(x,y)dxdy$ and $\iint\limits_R f(x,y)dydx$.

(b) One of the above ways is easier - use it to compute the volume. You must simplify your answer to a single fraction. (If you did it the easier way, simplifying will not be that bad).

Hint: Is there any nice symmetry?

a) First,



and $x=y^2 \Rightarrow y = \pm\sqrt{x}$. Second, our difference in height is given by $(1+x^2y^2-1)$, like $z_{top} - z_{bot}$.

so, $V = \int_{x=0}^{x=4}\int_{y=-\sqrt{x}}^{y=\sqrt{x}} x^2y^2\,dy\,dx = \int_{y=-2}^{y=2}\int_{x=y^2}^{x=4} x^2y^2\,dx\,dy$

+1 +1

b) $\int_0^4\int_{-\sqrt{x}}^{\sqrt{x}} x^2y^2\,dy\,dx$

$= \int_0^4 \left.\frac{x^2y^3}{3}\right|_{-\sqrt{x}}^{\sqrt{x}} dx$ +1

symetry is $x^{7/2}$

$\stackrel{\text{symetry}}{=} \int_0^4 \frac{2x^2\cdot x^{3/2}}{3}\,dx$

$= \left.\frac{2}{3}\cdot x^{9/2}\cdot\frac{2}{9}\right|_{x=0}^{x=4}$ +1

$= \frac{2\cdot 4^{9/2}\cdot 2}{27} = \boxed{\frac{4^{11/2}}{27}}$

+1

I think this was easier

OR

$\int_{-2}^{2}\int_{y^2}^{4} x^2y^2\,dx\,dy$

$= \int_{-2}^{2} \left.\frac{y^2x^3}{3}\right|_{x=y^2}^{x=4} dy$

$= \int_{-2}^{2} \frac{64y^2-y^8}{3}\,dy$ +1

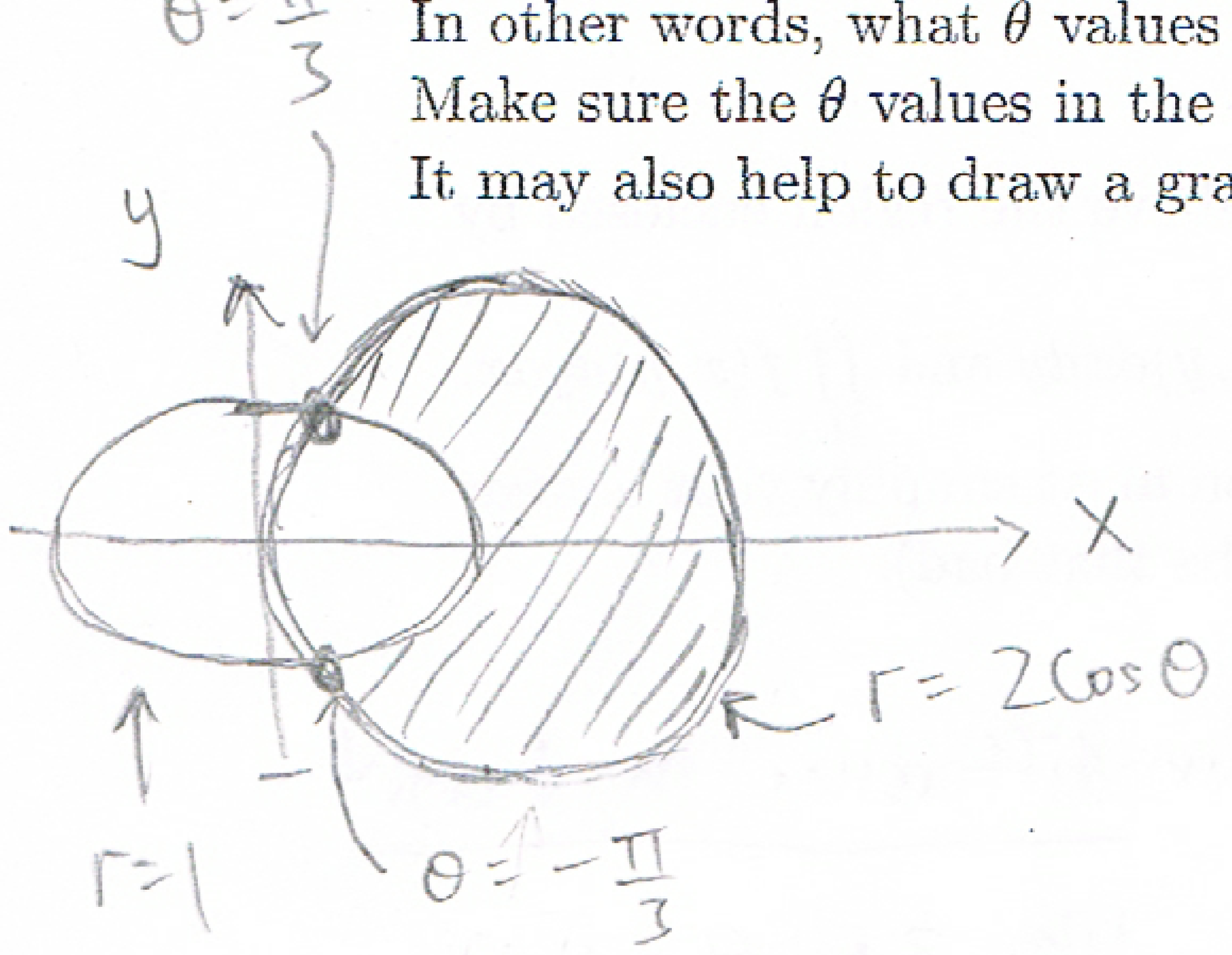
$\stackrel{\text{symmetry}}{=} 2\int_0^2 \frac{64y^2-y^8}{3}\,dy$. +1

$= 2\left[\frac{64y^3}{9} - \frac{y^9}{27}\right]\Big|_{y=0}^{y=2}$ ☆ $64 = 2^6$

$= 2\left[\frac{2^9}{9} - \frac{2^9}{27}\right] = 2\left[\frac{3\cdot 2^9 - 2^9}{27}\right]$

$= \frac{2\cdot 2\cdot 2^9}{27} = \boxed{\frac{2^{11}}{27}}$ +1 (ie. inside $(x-1)^2+y^2=1$ but outside $x^2+y^2=1$)

2. Find the area inside the circle $r = 2\cos\theta$ *but* outside the circle $r = 1$.
Hint: To get the range of θ to integrate over, you need the θ values at which the two circles intersect.
In other words, what θ values make the two circles equal?
Make sure the θ values in the upper and lower limits of your integral make sense.
It may also help to draw a graph of this.



- Intercepts: $2\cos\theta = 1$ (set r's equal)

  $\cos\theta = \frac{1}{2} \rightarrow \theta = \pm\frac{\pi}{3}$ +1

- We have to "scoop out" the inner circle.

or, see that r goes from 1 to $2\cos\theta$.

So, $A = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_{r=1}^{r=2\cos\theta} r\,dr\,d\theta = \underbrace{\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_{0}^{2\cos\theta} r\,dr\,d\theta}_{\text{Inside } r=2\cos\theta} - \underbrace{\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_{0}^{1} r\,dr\,d\theta}_{\text{scoop out what's inside } r=1.}$

(Either Approach is good) +2

$$A = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_{r=1}^{r=2\cos\theta} r\,dr\,d\theta = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{r^2}{2}\Big|_{r=1}^{r=2\cos\theta} d\theta$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(2\cos^2\theta - \frac{1}{2}\right) d\theta \quad +1$$

☆ $2\cos^2\theta = \cancel{2}\left(\cancel{\tfrac{1}{2}}(1+\cos 2\theta)\right) = 1+\cos 2\theta$.

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(\frac{1}{2} + \cos 2\theta\right) d\theta$$

symetry

$$\stackrel{\text{symetry}}{=} \cancel{2}\left(\frac{\theta}{\cancel{2}} + \frac{\sin 2\theta}{\cancel{2}}\right)\Big|_{\theta=0}^{\theta=\pi/3} = \frac{\pi}{3} + \sin\left(\frac{2\pi}{3}\right) \quad +1$$

$$= \boxed{\frac{\pi}{3} + \frac{\sqrt{3}}{2}}$$

Comment: Some got $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$ and had $\int_{\frac{\pi}{3}}^{5\pi/3}$.

This gets: ☹